Metric Temporal Logic Translations over the Naturals

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Abstract: We study translations from Metric Temporal Logic (MTL) over the natural numbers to Linear Temporal Logic (LTL). In particular, we present two approaches for translating from MTL to LTL which preserve the EXPSPACE complexity of the satisfiability problem for MTL. Our translations, thus, allow us to utilise LTL provers to solve MTL satisfiability problems.

1 Introduction

Linear and branching-time temporal logics have been used for the specification and verification of reactive systems. In linear-time temporal logic [4] we can, for example, express that a formula ψ holds now or at some point in the future using the formula $\Diamond\psi$ (ψ holds eventually). However, some applications require not just that a formula ψ will hold eventually but that it holds within a particular time-frame for example between 3 and 7 moments from now. To express metric constraints, a range of Metric Temporal Logics (MTL) have been proposed, considering different underlying models of time and operators allowed [3]. Here we use MTL with pointwise discrete semantics, following [1], where each state in the sequence is mapped to a time point on a time line isomorphic to the natural numbers. In this instance of MTL, temporal operators are annotated with certain finite as well as infinite intervals, for example, $\Box_{[2,4]}p$ means that p should hold in all states that occur between the interval $[2,4]$ of time, while $\Box_{[2,\infty)}p$ means that p should hold in all states that occur at least 2 moments from now. We provide two satisfiability preserving translations from MTL into LTL. Both translations are polynomial in the size of the MTL formula and the largest constant occurring in an interval (although exponential in the size of the MTL formula due to the binary encoding of the constants). Since the satisfiability problem for LTL is PSPACE [5], our translations preserve the EXPSPACE complexity of the MTL satisfiability problem [1].

2 Metric Temporal Logic Translations

We briefly state the syntax and semantics of LTL and MTL. Let $\mathcal{P}$ be a (countably infinite) set of propositional symbols. Well formed formulae in LTL are formed according to the rule: $\varphi, \psi := p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \bigcirc\varphi \mid (\varphi\mathcal{U}\psi)$ where $p \in \mathcal{P}$.

LTL Semantics. A *state sequence* σ over $(\mathbb{N}, <)$ is an infinite sequence of states $\sigma_i \subseteq \mathcal{P}$, $i \in \mathbb{N}$.

$(\sigma, i) \models p$	iff $p \in \sigma_i$
$(\sigma, i) \models (\varphi \wedge \psi)$	iff $(\sigma, i) \models \varphi$ and $(\sigma, i) \models \psi$
$(\sigma, i) \models \neg\varphi$	iff $(\sigma, i) \not\models \varphi$
$(\sigma, i) \models \bigcirc\varphi$	iff $(\sigma, i+1) \models \varphi$
$(\sigma, i) \models (\varphi\mathcal{U}\psi)$	iff there is $k \geq i$ such that $(\sigma, k) \models \psi$ and for all $j \in \mathbb{N}$, if $i \leq j < k$ then $(\sigma, j) \models \varphi$

We denote by $\bigcirc^c$ a sequence of c next operators. Further connectives can be defined as usual: $p \vee \neg p \equiv \mathbf{true}$, $\mathbf{true} \equiv \neg(\mathbf{false})$, $\mathbf{true}\mathcal{U}\varphi \equiv \Diamond\varphi$ and $\Diamond\varphi \equiv \neg\Box\neg\varphi$. MTL formulae are constructed in a way similar to LTL, with the difference that temporal operators are now bounded by an interval I with natural numbers as end-points or ∞ on the right side. Note that since we work with natural numbers as end-points we can assume w.l.o.g that all our intervals are of the form $[c_1, c_2]$ or $[c_1, \infty)$, where $c_1, c_2 \in \mathbb{N}$. Well formed formulae in MTL are formed according to the rule: $\varphi, \psi := p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \bigcirc_I\varphi \mid (\varphi\mathcal{U}_I\psi)$ where $p \in \mathcal{P}$.

MTL Semantics. A *strict timed state sequence* $\rho = (\sigma, \tau)$ over $(\mathbb{N}, <)$ is a pair consisting of an infinite sequence σ of states $\sigma_i \subseteq \mathcal{P}$, $i \in \mathbb{N}$, and a function $\tau : \mathbb{N} \to \mathbb{N}$ that maps every i corresponding to the i-th state to a time point $\tau(i)$ such that $\tau(i) < \tau(i+1)$.

$(\rho, i) \models \bigcirc_I\varphi$	iff $(\rho, i+1) \models \varphi$ and $\tau(i+1) - \tau(i) \in I$
$(\rho, i) \models (\varphi\mathcal{U}_I\psi)$	iff there is $k \geq i$ such that $\tau(k) - \tau(i) \in I$ and $(\rho, k) \models \psi$ and for all $j \in \mathbb{N}$, if $i \leq j < k$ then $(\rho, j) \models \varphi$

We omit propositional cases, which are as in LTL. Further connectives can be defined as usual: $\mathbf{true}\mathcal{U}_I\varphi \equiv \Diamond_I\varphi$ and $\Diamond_I\varphi \equiv \neg\Box_I\neg\varphi$. To transform an MTL formula into Negation Normal Form, one uses the constrained dual until $\tilde{\mathcal{U}}_I$ operator [3], defined as $(\varphi\tilde{\mathcal{U}}_I\psi) \equiv \neg(\neg\varphi\mathcal{U}_I\neg\psi)$. An MTL formula φ is in *Negation Normal Form (NNF)* iff the negation operator ($\neg$) occurs only in front of propositional variables. An MTL formula φ is in *Flat Normal Form (FNF)* iff it is of the form $p_0 \wedge \bigwedge_i \Box_{[0,\infty)}(p_i \rightarrow \psi_i)$ where p_0, p_i are propositional variables or $\mathbf{true}$ and ψ_i is either a formula of propositional logic or it is of the form $\bigcirc_I\psi_1$, $\psi_1\mathcal{U}_I\psi_2$ or $\psi_1\tilde{\mathcal{U}}_I\psi_2$ where ψ_1, ψ_2 are formulae of propositional logic. The transformations into NNF and FNF are satisfiability preserving and can be performed in polynomial time. From now on assume that our MTL formulae are in NNF and FNF.

From MTL to LTL: encoding 'gaps' We translate MTL formulae for discrete time models into LTL using a new propositional symbol gap. $\neg gap$ is true in those states σ'_j of σ' such that there is $i \in \mathbb{N}$ with $\tau(i) = j$ and gap is true in all other states of σ'. As shown in Table 1, we translate for example $\bigcirc_{[2,3]}p$ into: $\bigvee_{2 \leq l \leq 3}(\bigcirc^l(\neg gap \wedge p) \wedge$

MTL	LTL Gap Translation
$(\bigcirc_{[c_1,\infty)}\alpha)^\sharp$	$(\bigwedge_{1\leq k<c_1}\bigcirc^k gap)$ $\wedge \bigcirc^{c_1}(gap\mathcal{U}(\alpha\wedge\neg gap))$
$(\bigcirc_{[c_1,c_2]}\alpha)^\sharp$	$\bigvee_{c_1\leq l\leq c_2}(\bigcirc^l(\neg gap\wedge\alpha)$ $\wedge \bigwedge_{1\leq k<l}\bigcirc^k gap)$
$(\alpha\mathcal{U}_{[c_1,\infty)}\beta)^\sharp$	$(\bigwedge_{0\leq k<c_1}\bigcirc^k(gap\vee\alpha))$ $\wedge\bigcirc^{c_1}((gap\vee\alpha)\mathcal{U}(\neg gap\wedge\beta))$
$(\alpha\mathcal{U}_{[c_1,c_2]}\beta)^\sharp$	$\bigvee_{c_1\leq l\leq c_2}(\bigcirc^l(\neg gap\wedge\beta)$ $\wedge\bigwedge_{0\leq k<l}\bigcirc^k(gap\vee\alpha))$

Table 1: MTL to LTL translation using 'gap' where α, β are propositional logic formulae and $c_1, c_2 > 0$.

$\bigwedge_{1\leq k<l}\bigcirc^k gap)$.

Theorem 1. Let $\varphi = p_0 \wedge \bigwedge_i \Box_{[0,\infty)}(p_i \rightarrow \psi_i)$ be an MTL formula in NNF and FNF. Let $\varphi^\sharp = p_0 \wedge \bigwedge_i \Box(p_i \rightarrow (\neg gap \wedge \psi_i^\sharp))$ be the result of replacing each ψ_i in φ by $\psi_i^\sharp$ as in Table 1. Then, φ is satisfiable if, and only if, $\varphi^\sharp \wedge \neg gap \wedge \Box(\Diamond\neg gap)$ is satisfiable.

From MTL to LTL: encoding time differences Let $C-1$ be the greatest number occurring in an interval in an MTL formula φ or 1, if none occur. W.l.o.g., we can consider only strict timed state sequences where the time difference from a state to its previous state is bounded by C [2]. Then, we can encode time differences with a set $\Pi_\delta = \{\delta_i^- \mid 1 \leq i \leq C\}$ of propositional variables where each δ_i^- represents a time difference of i w.r.t. the previous state (one could also encode the time difference to the next state instead of the difference from the previous state). We also encode variables of the form s_m^n with the meaning that 'the sum of the time differences from the last n states to the current state is m'. For our translation, we only need to define these variables up to sums bounded by $2\cdot C$.

To simplify the presentation, we use two additional n-ary boolean operators $\oplus_{=1}$ and $\oplus_{\leq 1}$. If $S = \{\varphi_1,\ldots,\varphi_n\}$ is a finite set of LTL formulae, then $\oplus_{=1}(\varphi_1,\ldots,\varphi_n)$, also written $\oplus_{=1}S$, is a LTL formula. Let σ' be a state sequence and $i\in\mathbb{N}$. Then $(\sigma',i)\models\oplus_{=1}S$ iff $(\sigma',i)\models\varphi_j\in S$ for exactly one $\varphi_j\in S$, $1\leq j\leq n$. Similarly, $(\sigma',i)\models\oplus_{\leq 1}S$ iff $(\sigma',i)\models\varphi_j\in S$ for at most one $\varphi_j\in S$, $1\leq j\leq n$. Let S_C be the conjunction of the following:

1. $\bigcirc\Box\oplus_{=1}\Pi_\delta$, for $\Pi_\delta = \{\delta_k^- \mid 1\leq k\leq C\}$;
2. $\Box(\delta_k^- \leftrightarrow s_k^1)$, for $1\leq k\leq C$;
3. $\Box\oplus_{\leq 1}\Pi^i$, for $1\leq i\leq 2\cdot C$ and $\Pi^i = \{s_j^i \mid i\leq j\leq 2\cdot C\}$;
4. $\Box(\bigcirc s_k^1 \wedge s_l^j \rightarrow \bigcirc s_{l+k}^{j+1})$, for $1 < j+1 \leq l+k \leq 2\cdot C$.

Point 1 ensures that at all times, the time difference k from the current state to the previous (if it exists) is uniquely encoded by the variable δ_k^-. In Point 2 we have that the sum of the difference of the last state to the current, encoded by s_k^1, is exactly δ_k^-. Point 3 ensures that at all times we cannot have more than one value for the sum of the time differences of the last i states. Finally, Point 4 has the propagation of sum variables: if the sum of the last j states is l and the time difference to the next is k then the next state should have that the sum of the last $j+1$ states is $l+k$. As shown in Table 2, we translate, for example, $\bigcirc_{[2,3]}p$ into $\bigcirc((\delta_2^-\vee\delta_3^-)\wedge p)$.

MTL	LTL Time Difference Translation
$(\bigcirc_{[c_1,\infty)}\alpha)^\sharp$	$\bigcirc((\bigvee_{c_1\leq i\leq C}\delta_i^-)\wedge\alpha)$
$(\bigcirc_{[c_1,c_2]}\alpha)^\sharp$	$\bigcirc((\bigvee_{c_1\leq i\leq c_2}\delta_i^-)\wedge\alpha)$
$(\alpha\mathcal{U}_{[c_1,\infty)}\beta)^\sharp$	$\bigvee_{1\leq i\leq c_1}(\bigcirc^i((\bigvee_{c_1\leq j\leq c_1+C} s_j^i)\wedge\alpha\mathcal{U}\beta)$ $\wedge(\bigwedge_{0\leq k<i}\bigcirc^k\alpha))$
$(\alpha\mathcal{U}_{[c_1,c_2]}\beta)^\sharp$	$\bigvee_{1\leq i\leq c_2}(\bigcirc^i((\bigvee_{c_1\leq j\leq c_2} s_j^i)\wedge\beta)$ $\wedge(\bigwedge_{0\leq k<i}\bigcirc^k\alpha))$

Table 2: MTL to LTL translation where α, β are propositional logic formulae and $c_1, c_2 > 0$.

Theorem 2. Let $\varphi = p_0 \wedge \bigwedge_i \Box_{[0,\infty)}(p_i \rightarrow \psi_i)$ be an MTL formula in NNF and FNF. Let $\varphi^\sharp = p_0 \wedge \bigwedge_i \Box(p_i \rightarrow \psi_i^\sharp)$ be the result of replacing each ψ_i in φ by $\psi_i^\sharp$ as in Table 2. Then, φ is satisfiable if, and only if, $\varphi^\sharp \wedge S_C$ is satisfiable.

3 Conclusion

We presented two translations from MTL to LTL. These translations provide a route to practical reasoning about MTL over natural numbers via LTL solvers. Our second translation and the MTL decision procedure presented in [1] are based on time differences and use the bounded model property. However, the translations using 'gap' do not require this property.

References

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