Modal Tableau Systems with Blocking and Congruence Closure

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Abstract: The interest of our research are semantic tableau approaches closely related to bottom-up model generation methods. Using equality-based blocking techniques these can be used to decide logics representable in first-order logic that have the finite model property. Many common modal and description logics have these properties and can therefore be decided in this way. This paper integrates congruence closure, which is probably the most powerful and efficient way to realise reasoning with ground equations, into a modal tableau system with equality-based blocking. The system is described for an extension of modal logic **K** characterised by frames in which the accessibility relation is transitive and every world has a distinct immediate predecessor. We show the system is sound and complete, and discuss how various forms of blocking such as ancestor blocking can be realised in this setting.

Tableau systems provide a natural and powerful form of reasoning widely used for non-classical logics, especially modal, description, and hybrid logics. In this work the focus is on semantic tableau systems closely related to bottom-up model generation methods [2]. Using unrestricted blocking [12], which is an equality-based blocking technique, these can decide logics with the finite model property, representable in first-order logic [13, 14]. Many common modal and description logics have these properties and can therefore be decided using semantic tableau systems with equality-based blocking.

For many common modal and description logics there are ways to avoid the explicit use of equality in the tableau system. For more expressive logics, with nominals as in hybrid modal logics and description logics (nominals are distinguished propositional variables that hold at exactly one world), it becomes harder to avoid the explicit handling of equality (though not impossible [7]). For modal logics where the binary relations satisfy frame conditions expressible as first-order formulae with equality, explicit handling of equations is the easiest and sometimes the only known way to perform equality reasoning. Single-valuedness of a relation is an example of a frame condition expressed using equality. Another example is the following

$$\forall x \exists y \forall z \big(R(y,x) \wedge x \not\approx y \wedge \big((R(y,z) \wedge R(z,x)) \rightarrow (z \approx x \vee z \approx y)\big)\big), \tag{1}$$

where $\approx$ denotes equality. Provision for explicit equality reasoning is also necessary for tableau systems with equality-based blocking.

In semantic tableau systems explicit equality handling has been realised in a variety of ways. Using standard equality rules is conceptually easiest and most general, and is often used [4, 5, 12]. This approach leads to a combinatorial explosion of derived formulae to ensure all elements in the same equivalence class have the same information content. Many of these formulae are unneeded and fewer formulae are derived when using paramodulation-style rules, where the central idea is replacement of equals by equals [3, 5]. Ordered rewriting presents a further refinement and is significantly more efficient because equations are oriented by an ordering and then used to simplify the formulae. Ordered rewriting is used, e.g., in a semantic tableau system of [9] for the description logic $\mathcal{SHOI}$. Different equality reasoning methods have also been integrated into non-ground tableau and related approaches, e.g. [3, 5, 6].

We require efficient handling of *ground equations*. For this purpose congruence closure algorithms provide probably the most efficient algorithms [10]. The Nelson-Oppen congruence closure method has been incorporated with Smullyan-type tableau system for first-order logic by [8]. Congruence closure algorithms have also been very successfully combined with the DPLL approach and are standardly integrated in SMT-solvers as theory reasoners for the theory of equations with uninterpreted function [11].

The motivation of the present work is to combine congruence closure with semantic tableau systems for modal, description, and hybrid logics. Since it presents a general framework in which many existing congruence closure algorithms can be described (and in order to achieve more generality), we combine the *abstract congruence closure system* of [1] with our semantic tableau system. Our ultimate goal is to provide a general framework with general soundness and completeness results for developing and studying equality reasoning and blocking in semantic tableau systems. The tableau system we consider has been obtained in the tableau synthesis framework of [13], but in this framework equality is accommodated by the standard equality rules. We have shown how these can be replaced by abstract congruence closure rules.

The most closely related work is the aforementioned [8], because the flavour of the tableau systems we are concerned with is similar to that of Smullyan-type tableau systems for first-order logic. The key difference is the way in which we use the congruence closure algorithm: In [8], the congruence closure component is essentially a black box that is queried to check entailed equalities. In contrast, we use the convergent term rewrite system produced by the abstract congruence closure algorithm also systematically to normalise the remaining tableau formulae. This means that duplication of formulae is avoided and that restrictions of the search space that depend on normalisation can be applied easily. In addition, we show that the ideas are not limited to a fixed set of the well-known tableau rules for first-order logic, but can be combined with special-purpose tableau systems of other logics having other kinds of tableau rules. Also related is [9] and the implementation of equality reasoning in METTEL-generated tableau provers [16], where ordered rewriting is used. This work does however not have the same level of generality as abstract congruence closure.

We present an abstract semantic tableau system with abstract ways of handling both blocking and equality. The focus is on showing how the abstract congruence closure system of [1] can be combined with a semantic tableau system for a modal logic. In contrast to earlier work, we use a "white box" integration, so that the abstract congruence closure is not only used to check entailed equalities, but also to normalize tableau formulae, so that logically equivalent formulae are eliminated. The particular modal tableau system was chosen to illustrate the most important ideas of integrating congruence closure so that the integration can be extended to other tableau systems for other modal, description, and hybrid logics. We believe the case study is general enough to work out how to combine congruence closure with Smullyan-type tableau rules for first-order logic, or incorporate it into bottom-up model generation and hypertableau methods. The ideas are also applicable in tableau systems obtained in the tableau synthesis framework of [13].

The paper in which this work is publieshed is [15].

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