Towards Generalised Proof Search for Natural Deduction Systems for logics $I_{\langle\alpha,\beta\rangle}$

Alexander Bolotov[1] Vasily Shangin[2*]

[1] University of Westminster, Department of Computer Science A.Bolotov@wmin.ac.uk
[2] Lomonosov Moscow State University, Faculty of Philosophy, Department of Logic, shangin@philos.msu.ru

Abstract: We continue our investigation of the proof searching procedures developed for natural deduction calculus for classical and a variety of non-classical logics. In particular, we deal with natural deduction systems for propositional logics $I_{\langle\alpha,\beta\rangle}$, where $\alpha, \beta \in 0, 1, 2, 3, \ldots \omega$ such that $I_{\langle 0,0\rangle}$ is classical logic, proposed by Vladimir Popov. We aim at generalising the concept of an inference for these systems that is fundamental to proof searching technique for these logics.

1 Introduction

In [6], a logic $I_{\langle\alpha,\beta\rangle}$ is proposed as Hilbert-style calculi $HI_{\langle\alpha,\beta\rangle}$, where $\alpha, \beta \in 0, 1, 2, 3, \ldots \omega$ such that $HI_{\langle 0,0\rangle}$ is classical logic, to deal with a generalization of Glivenko theorem [5]. We present natural deduction calculi $ND_{\langle\alpha,\beta\rangle}$, for these logics. We show that A is a theorem of $HI_{\langle\alpha,\beta\rangle}$ iff A is a theorem of $ND_{\langle\alpha,\beta\rangle}$. Moreover, we present a generalised proof search technique for each natural deduction calculus in question. The propositional language L over the alphabet $p, p_1, p_2, \ldots, (,), Bool$ ($Bool = \wedge, \supset, \vee, \neg$) and a notion of a formula of language L are defined in the standard way. A formula is said to be quasi-elementary iff no logical connective of $Bool$ occurs in it [6]. Let $|A|$ abbreviate the length of A, the number of all occurrences of the logical connectives of L in A. Letters A, B, C, D, E with lower indices run over arbitrary formulae. Letters α, β with upper and lower indices run over arbitrary finite sets of formulae. Letters α, β run over $0, 1, 2, 3, \ldots \omega$.

2 Hilbert-style systems $HI_{\langle\alpha,\beta\rangle}$

In [6], V. Popov presents a Hilbert-style calculus $HI_{\langle\alpha,\beta\rangle}$ with the following axioms:

(I) $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

(II) $A \supset (A \vee B)$

(III) $B \supset (A \vee B)$

(IV) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$

(V) $(A \wedge B) \supset A$

(VI) $(A \wedge B) \supset B$

(VII) $(C \supset A) \supset ((C \supset B) \supset (C \supset (A \wedge B)))$

(VIII) $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$

(IX) $((A \wedge B) \supset C) \supset (A \supset (B \supset C))$

(X) $(((A \supset B) \supset A) \supset A)$

(XI) $\neg D \supset (D \supset A)$, where D is not a quasi-elementary formulae and $|D| < \alpha$

(XII $(E \supset (\neg A \supset A)) \supset E$, where E is not a quasi-elementary formulae and $|E| < \beta$.

Modus ponens is the only inference rule of the calculus. Definitions of an inference and proof in $HI_{\langle\alpha,\beta\rangle}$ are standard as well as notions of their length.

3 ND systems $ND_{\langle\alpha,\beta\rangle}$

$$\wedge\, el_1 \ \frac{A \wedge B}{A} \qquad \wedge\, el_2 \frac{A \wedge B}{B} \qquad \wedge\, in \frac{A,\ B}{A \wedge B}$$

$$\vee\, in_1 \ \frac{A}{A \vee B} \qquad \vee\, in_2 \frac{B}{A \vee B} \qquad \supset el \frac{A \supset B,\ A}{B}$$

$$\neg\, in_{1_\alpha} \ \frac{D, \neg D}{E}$$ where D is not a quasi-elementary formula with $|D| < \alpha$.

$$\supset\, in \ \frac{[A]B}{A \supset B} \qquad \supset_p \frac{[A \supset B]\ A}{A}$$

$$\neg\, in_{2_\beta} \frac{[E]\ \neg(B \supset B)}{\neg E}$$

Formulae in the square brackets are the last in the list of assumptions. Additionally, in $\neg\ in_{2_\beta}$, formula E is not a quasi-elementary formula with $|E| < \beta$.

An inference is said to be a non-empty finite linearly ordered sequence of formulae $C_1, C_2, \ldots, C_k$, satisfying the following conditions:

- Each C_i is either an assumption or is inferred from the previous formulae via an ND rule;
- In applying $\supset\ in$, each formula, starting from the last assumption $[A]$ up to (but not including) $A \supset B$, the result of this rule, is discarded from the inference;

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- In applying $\supset_P$, each formula, starting from the last premise $[A \supset B]$ up to (but not including) A, the result of this rule, is discarded from the inference;
- In applying $\neg\ in2_\beta$,each formula, starting from the last premise $[E]$ up to (but not including) E, the result of this rule, is discarded from the inference.

Given an inference $C_1, C_2, \ldots, C_k$ with $A_1, A_2, \ldots, A_n$ being non-discarded assumptions and with the last formula $C_k = B$, we have an inference of B from assumptions $A_1, A_2, \ldots, A_n$. If the set of formulae Γ contains $A_1, A_2, \ldots, A_n$ and there is an inference of B from $A_1, A_2, \ldots, A_n$ then we say there is an inference of B from a set of formulae Γ [1].

MAIN THEOREM

$\Gamma \vdash_{HI_{\langle\alpha,\beta\rangle}} A \iff \Gamma \vdash_{ND_{\langle\alpha,\beta\rangle}} A$, for each $\alpha, \beta \in \{0, 1, 2, 3, \ldots \omega\}$.

Proof Idea. Left to Right:

We need to show that if there exists an inference of A from Γ in $HI_{\langle\alpha,\beta\rangle}$ then there exists an inference of A from Γ in $ND_{\langle\alpha,\beta\rangle}$. For the proof we define a notion of a "height of an inference" (similar to the definitions of heights of proofs, so that the height of the inference of an axiom is 1, etc) and then we prove this direction of the theorem by mathematical induction on the height of the inference of an arbitrary formula A from Γ in $HI_{\langle\alpha,\beta\rangle}$. The base case would require to prove all axioms in the ND system. The proof for the inductive step is of course more involved and it uses the structural similarity of the modus ponens rule in the axiomatics and an ND $\supset\ el$ rule.

Right to Left:

We need to show that if there exists an inference of A from Γ in $ND_{\langle\alpha,\beta\rangle}$ then there exists an inference of A from Γ in $HI_{\langle\alpha,\beta\rangle}$. We note that for the base case the ND inferences are "trivial" and it is easy to construct corresponding axiomatic proof. For the inductive step, the proof is complex and is based on the identification of the cases of the applications of the ND rules.

4 Towards Generalised Proof Search for $ND_{\langle\alpha,\beta\rangle}$

Here we draw a route to formulating this generalised proof search for natural deduction calculi. We first note that proof search for various logics is based on the notion of algo-derivation that is served to establish inferences in an automated way (for decidable logics).

Algo-derivation in $ND_{\langle\alpha,\beta\rangle}$, abbreviated as $ND_{\langle\alpha,\beta\rangle_{\mathsf{ALG}}}$, is a pair (*list_proof*, *list_goals*) whose construction is determined by the searching procedure outlined below.

Below we give a very short insight into the searching procedures referring the reader to [4], [3], [2] for more detailed description of various searching techniques that formed "classical" propositional reasoning in natural deduction representations of linear-time temporal logic, paraconsistent logic PCont and paracomplete logic PComp.

Searching Procedures.

Procedure (1). Here we search for an applicable elimination ND-rule in order to update *list_proof*.

Procedure (2). We look at the structure of the current goal and update *list_proof* and *list_goals*, respectively, by new goals or new assumptions. If no updates are possible and the current goal is not reached we analyse compound formulae in *list_proof* in order to find sources for new goals.

Procedure (3). This checks the reachability of the current goal in the sequence *list_goals*.

Procedure (4). Procedure (4) results in finding a relevant introduction rule to be applied. As we have already noted, the specifics of our searching technique is complete determination of the application of the introduction rules. Any application of such a rule is strictly determined by the current goal in *list_goals*.

CONJECTURE

Abbreviating an algo-proof of A from Γ in $ND_{\langle\alpha,\beta\rangle}$ by $\Gamma \vdash_{ND_{\langle\alpha,\beta\rangle_{\mathsf{ALG}}}} A$, we aim to establish the following:

For for each $\alpha, \beta \in \{0, 1, 2, 3, \ldots \omega\}$, $\Gamma \vdash_{ND_{\langle\alpha,\beta\rangle}} A$ if, and only if, there exists $\Gamma \vdash_{ND_{\langle\alpha,\beta\rangle_{\mathsf{ALG}}}} A$.

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